ON THE GENERATING HYPOTHESIS IN NONCOMMUTATIVE STABLE HOMOTOPY

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Abstract. Freyd's Generating Hypothesis is an important problem in topology with deep structural consequences for finite stable homotopy. Due to its complexity some recent work has examined analogous questions in various other triangulated categories. In this short note we analyze the question in noncommutative stable homotopy, which is a natural extension of finite stable homotopy.

Introduction

In (finite) stable homotopy theory the Spanier–Whitehead category of finite spectra, denoted by $\mathtt{SW}^f$, is a central object of study. Roughly speaking, it is constructed by formally inverting the suspension functor on the category of finite pointed CW complexes and it is a triangulated category. Its counterpart in the noncommutative setting is the triangulated noncommutative stable homotopy category, denoted by $\mathfrak{S}$. This triangulated category was constructed by Thom [20] (see also [7]) building upon earlier works of Rosenberg [17], Connes–Higson [5], Dădărlat [6] and Houghton-Larsen–Thomsen [12] amongst others. The triangulated category $\mathfrak{S}$ is a canonical generalization of $\mathtt{SW}^f$. It comes in a mysterious package carrying vital information about bivariant homology theories on the category of separable C^*-algebras.

Freyd stated the following *Generating Hypothesis* in [11] (Chapter 9):

Conjecture (Freyd)**.** The object $(S^0, 0)$ is a graded generator in $\mathtt{SW}^f$, where S^0 is the pointed 0-sphere.

An alternative formulation of the Generating Hypothesis asserts that for any two finite spectra X, Y the canonical homomorphism

$$\Phi : \mathtt{SW}^f(X,Y) \to \mathrm{Hom}_{\pi_*(\mathbb{S})}(\pi_*(X), \pi_*(Y)) \tag{1}$$

is injective. Here $\mathbb{S}$ denotes the sphere spectrum and $\mathtt{Mod}(\pi_*(\mathbb{S}))$ denotes the category of right modules over the graded commutative ring $\pi_*(\mathbb{S})$. The conjecture has some interesting reformulations and generalizations [11, 8, 13, 3, 19]. It remains an open problem at the time of writing this article. However, it has spurred a lot of stimulating research. Analogues of the Generating Hypothesis have been addressed in several other contexts, such as the stable module category of a finite group algebra [1, 4], the derived category of a ring [15, 14], equivariant stable homotopy [2], and so on.

By Proposition 9.7 of [10] (see also Corollary 3.2 of [13]) the injectivity of the map Φ above automatically implies its bijectivity. If true, the Generating Hypothesis would reduce

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the task of understanding the stable homotopy classes of maps between finite pointed CW complexes to a more tractable algebraic problem, i.e., understanding the module category of $\pi_*(\mathbb{S})$.

In the first part of this article (Section 1) we show that the naïve generalization of the Generating Hypothesis to noncommutative stable homotopy fails to hold. More precisely, we show

Theorem. The corresponding map $\Phi : \mathfrak{S}(A, B) \to \mathrm{Hom}_{\pi_*(\mathbb{C})}(\pi_*(A), \pi_*(B))$ in $\mathfrak{S}$ is neither injective nor surjective in general.

In the second part of the paper (Section 2) we explain the issues that the naïve generalization suffers from. The main problem is the *size* of $\mathfrak{S}$ and we provide a modified formulation by restricting our attention to a suitable subcategory (see Question 2.2), which implies Freyd's Generating Hypothesis.

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1. Naïve Generating Hypothesis in $\mathfrak{S}$

Let $\mathtt{SC}^*$ denote the category of separable C^*-algebras and $*$-homomorphisms. Recall from [20] that there is a stable homotopy functor $\pi_* : \mathtt{SC}^* \longrightarrow \mathtt{Mod}(\pi_*(\mathbb{C}))$ in noncommutative topology, which factors through $\mathfrak{S}$ giving rise to the canonical map

$$\Phi : \mathfrak{S}(A, B) \to \mathrm{Hom}_{\pi_*(\mathbb{C})}(\pi_*(A), \pi_*(B)). \tag{2}$$

Here $\mathtt{Mod}(\pi_*(\mathbb{C}))$ denotes the category of right modules over the ring $\pi_*(\mathbb{C})$.

1.1. **Failure of injectivity of Φ.** It is known that on the category of stable C^*-algebras noncommutative stable homotopy agrees with bivariant E-theory [5, 6], i.e., $\mathfrak{S}(A, B) \cong \mathrm{E}_0(A, B)$ and $\pi_*(A) \cong \mathrm{E}_*(A)$. In addition, on the category of nuclear C^*-algebras one has $\mathrm{E}_0(A, B) \cong \mathrm{KK}_0(A, B)$ and $\mathrm{E}_*(A) \cong \mathrm{K}_*(A)$. The Universal Coefficient Theorem in KK-theory [18] asserts that there is a natural short exact sequence in the bootstrap class

$$0 \to \mathrm{Ext}^1(\mathrm{K}_*(\Sigma A), \mathrm{K}_*(B)) \to \mathrm{KK}_*(A, B) \to \mathrm{Hom}(\mathrm{K}_*(A), \mathrm{K}_*(B)) \to 0, \tag{3}$$

which splits unnaturally.

Proposition 1.1. The canonical map $\Phi : \mathfrak{S}(A, B) \to \mathrm{Hom}_{\pi_*(\mathbb{C})}(\pi_*(A), \pi_*(B))$ is not injective in general.

Proof. Let us choose judiciously $A = \mathrm{C}(X, x)\hat{\otimes}\mathbb{K}$ and $B = \mathrm{C}(Y, y)\hat{\otimes}\mathbb{K}$ ($\mathbb{K}$ being the algebra of compact operators) in such a manner that $\mathrm{Ext}^1(\mathrm{K}_*(\Sigma A), \mathrm{K}_*(B))$ is non-zero (with every group in (3) finitely generated). This can be easily achieved by choosing finite pointed CW complexes (X, x) and (Y, y), such that their K-theory groups contain torsion. Then $\mathfrak{S}(A, B) \cong \mathrm{KK}_0(\mathrm{C}(X, x), \mathrm{C}(Y, y))$ due to the C^*-stability of KK-theory in both variables. Now the map $\mathrm{KK}_0(A, B) \to \mathrm{Hom}(\mathrm{K}_*(A), \mathrm{K}_*(B))$ is not injective due to the non-vanishing of the Ext^1-term in (3), whence Φ cannot be injective since $\mathrm{Hom}_{\pi_*(\mathbb{C})}(\pi_*(A), \pi_*(B)) \subset \mathrm{Hom}(\mathrm{K}_*(A), \mathrm{K}_*(B))$. ∎

1.2. **Failure of the surjectivity of Φ.**

Lemma 1.2. $\mathfrak{S}(M_2(\mathbb{C}), \mathbb{C}) \cong 0$.

Proof. It is known that every element in $\mathfrak{S}(M_2(\mathbb{C}), \mathbb{C})$ can be represented as the homotopy class of a $*$-homomorphism $f : \Sigma^r M_2(\mathbb{C}) \to \Sigma^r \mathbb{C}$. Since $\Sigma^r \mathbb{C}$ is commutative and $M_2(\mathbb{C})$ is simple, f must be the zero morphism. □

Proposition 1.3. The canonical map $\Phi : \mathfrak{S}(A, B) \to \mathrm{Hom}_{\pi_*(\mathbb{C})}(\pi_*(A), \pi_*(B))$ is not surjective in general.

Proof. Set $A = M_2(\mathbb{C})$ and $B = \mathbb{C}$. By the previous Lemma it suffices to show that $\mathrm{Hom}_{\pi_*(\mathbb{C})}(\pi_*(A), \pi_*(B))$ is non-zero. Since $\pi_*(\mathbb{C})$ is isomorphic to the stable cohomotopy of spheres, one has $\pi_0(\mathbb{C}) \simeq \mathbb{Z}$ and the all the higher stable homotopy groups are torsion. Thus it suffices to show that $\pi_0(M_2(\mathbb{C}))$ contains a non-zero element of infinite order, which is clear. □

Remark 1.4. Due to the contravariance of the functor sending a space to its complex C^*-algebra, the Generating Hypothesis in $\mathfrak{S}$ should be viewed as an analogue of the Cogenerating Hypothesis in $\mathtt{SW}^f$, which is equivalent to the Generating Hypothesis in $\mathtt{SW}^f$ by the Spanier–Whitehead duality. One also sees above that $\mathfrak{S}$ is not equivalent to its opposite category.

2. Modified Generating Hypothesis

We called the Generating Hypothesis in the previous section *naïve* for the following reasons:

(1) In $\mathtt{SW}^f$ the objects are finite pointed CW complexes, whereas in $\mathfrak{S}$ they are arbitrary pointed compact metrizable spaces. The presence of finite matrix algebras in the noncommutative analogue of $\mathtt{SW}^f$ is non-negotiable, since they form natural building blocks of noncommutative CW complexes [9, 16]. However, $\mathbb{K}$ can be expressed as $\varinjlim_n M_n(\mathbb{C})$ in $\mathtt{SC}^*$, i.e., it can be regarded as a countable inverse limit of *noncommutative or fat points*. Thus $\mathbb{K}$ is not indispensable.

(2) We did not take finite matrix algebras into consideration as test objects for the Generating Hypothesis, which would be the correct way to think about the Generating Hypothesis in this situation [3].

Let us first address (1). The Spanier–Whitehead category is a tensor triangulated category under smash product of finite spectra. Thom proved in [20] that $\mathfrak{S}$ is a tensor (under maximal C^*-tensor product $\hat{\otimes}$) triangulated category.

Definition 2.1. We define the noncommutative Spanier–Whitehead category, denoted by $\mathfrak{SW}$, to be the smallest thick tensor triangulated subcategory of $\mathfrak{S}$ generated by $M_n(\mathbb{C})$ for all $n \in \mathbb{N}$.

Question 2.2 (Matrix Generating Hypothesis in $\mathfrak{SW}$)**.** Let $f \in \mathfrak{SW}(A, B)$ be a morphism, such that $\mathfrak{SW}(M_n(\mathbb{C}), f) : \mathfrak{SW}(M_n(\mathbb{C}), (A, i)) \to \mathfrak{SW}(M_n(\mathbb{C}), (B, i))$ is the zero morphism for all $n \in \mathbb{N}$ and $i \in \mathbb{Z}$. Is f itself the zero morphism in $\mathfrak{SW}(A, B)$?

The above formulation uses $M_n(\mathbb{C})$ for all $n \in \mathbb{N}$ as test objects and hence addresses (2). The full subcategory of $\mathfrak{SW}$ consisting of commutative C^*-algebras is equivalent to the (opposite of) $\mathtt{SW}^f$.

Proposition 2.3. The Matrix Generating Hypothesis in $\mathfrak{SW}$ implies Freyd's Generating Hypothesis in $\mathtt{SW}^f$.

Proof. Let $A = \mathrm{C}(X, x)$ and $B = \mathrm{C}(Y, y)$ be separable commutative C^*-algebras and $f \in \mathfrak{SW}(A, B) \cong \mathtt{SW}^f((Y, y), (X, x))$ be any morphism, such that $\pi_*(f) = 0$. Since B is commutative, arguing as in Lemma 1.2 we deduce that $\mathfrak{SW}(M_n(\mathbb{C}), (B, i)) = 0$ for all $n > 1$ and $i \in \mathbb{Z}$. It follows that $\mathfrak{SW}(M_n(\mathbb{C}), f) = 0$ whence by the Matrix Generating Hypothesis $f = 0$. $\square$

Remark 2.4. Notice that Proposition 1.1 is no longer applicable to $\mathfrak{SW}$. If $\mathbb{K}$ were included in the generating set of Definition 2.1, the corresponding *Matrix + Compact Generating Hypothesis* would once again be falsified by Proposition 1.1.

One also observes from Proposition 1.3 that a *faithful implies full* type statement cannot hold in $\mathfrak{SW}$ (if the Matrix Generating Hypothesis turns out to be true).

Let us clarify that our results in Section 1 do not invalidate Freyd's Generating Hypothesis in the usual finite stable homotopy category. The failure of the injectivity (resp. the surjectivity of Φ) in the bigger category $\mathfrak{S}$ crucially exploited the presence of genuinely noncommutative C^*-algebras. We hope that the generalized perspective will shed some light on the original problem.

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